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COLORINGS OF PLANE GRAPHS WITH NO RAINBOW FACES

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We prove that each n-vertex plane graph with girth $g \geq 4$ admits a vertex coloring with at least $\lceil n/2 \rceil + 1$ colors with no rainbow face, i.e., a face in which all vertices receive distinct colors. This proves a conjecture of Ramamurthi and West. Moreover, we prove for plane graph with girth $g \geq 5$ that there is a vertex coloring with at least $\left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil$ if g is odd and $\left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil$ if g is even. The bounds are tight for all pairs of n and g with $g \geq 4$ and $n \geq 5g/2 - 3$.

1. Introduction

Colorings of graphs on surfaces and in particular of plane graphs under face-constraints have recently attracted a lot of attention. Two natural face constraints are that a face is not monochromatic and that a face is not rainbow. A face is *monochromatic* if its vertices all have the same color and it is *rainbow* if its vertices all have mutually distinct colors. In this paper, we study vertex colorings of plane graphs with no rainbow faces.

Zykov [21] introduced the notion of planar hypergraphs which was generalized by Kündgen and Ramamurthi [16] to face hypergraphs of graphs embedded in surfaces of higher genera. A *planar hypergraph* is a hypergraph whose bipartite incidence graph between the vertices and the edges is planar. Equivalently, there is a plane graph G such that for every hyperedge E there is a face in the graph G whose vertex set is E (but there might be faces with

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no corresponding hyperedges). A *face hypergraph* is a hypergraph whose hyperedges are vertex sets of all faces. Vertex colorings with no monochromatic face are ordinary colorings of face hypergraphs. Dvořák, the second and the third author [7] proved that a face hypergraph of a graph with no digon embedded in a surface of Euler's genus ε can be colored by $O(\sqrt[3]{\varepsilon})$ colors. The vertex coloring of plane graphs with both types of restrictions on faces have been studied in [6,13,15]. Penaud [17] proved that each plane graph has a vertex 2-coloring with neither monochromatic nor rainbow faces. A recent result of Diwan [5] implies that each plane graph with at least five vertices has a vertex coloring with 3 colors with neither monochromatic nor rainbow faces. Extremal problems involving rainbow patterns have also been studied, see [1,2,4,8,10–12,19].

We call a vertex coloring of a plane graph *valid* if no face is rainbow. The maximum number of colors used in a valid coloring of a plane graph G is denoted by $\chi_f(G)$. Note that this number need not to be the same for different embeddings of the same planar graph. Ramamurthi and West [18] proved the following inequality relating $\chi_f(G)$ to other graph parameters:

$$\chi_f(G) \geq \alpha(G) + 1 \geq \left\lceil \frac{n}{\chi(G)} \right\rceil + 1. \tag{1}$$

Moreover, the bounds $\lceil n/2 \rceil + 1$ and $\lceil n/3 \rceil + 1$ are tight for G being 2-chromatic and 3-chromatic, respectively. The bound $\lceil n/4 \rceil + 1$ is within one of being tight for 4-chromatic plane graphs. Grötzsch's theorem [9,20] states that an arbitrary triangle-free plane graph G is 3-colorable, and hence $\chi_f(G) \geq \lceil n/3 \rceil + 1$, where n is the number of vertices for triangle-free plane graphs G. However, Ramamurthi and West [18] conjectured that this bound is far from being sharp:

Conjecture 1. If G is an n-vertex triangle-free plane graph, $n \geq 4$, then

$$\chi_f(G) \geq \lceil n/2 \rceil + 1.$$

Note that the conjecture deals with simple graphs. Otherwise, the graph G' obtained from a connected triangle-free graph G by replacing each edge with a digon is again triangle-free and $\chi_f(G') = 1$. Ramamurthi and West [18] proved their conjecture for plane graphs of girth at least six. The second author [14] proved the conjecture for plane graphs of girth five.

In this paper, we prove the above conjecture (Corollary 1). More generally, we prove tight bounds on the numbers of colors in a vertex coloring with no rainbow faces for plane graphs with no short cycles (Theorems 3 and 4).

We first prove the lower bound in Section 2: If G is a plane graph with n vertices and girth at least g, $5 \leq g \leq n$, then $\chi_f(G) \geq \left\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \right\rceil$ if g is odd and $\chi_f(G) \geq \left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil$ if g is even (Theorem 3). The bounds can be slightly improved to $\chi_f(G) \geq \left\lceil \frac{g-3}{g-2}n + \frac{2}{g-2} \right\rceil$ under an additional assumption that the number of faces of G is even.

The corresponding upper bounds are proved in Section 3. We construct an n-vertex plane graph with girth g for which the above lower bound is best possible for all pairs of g and n such that $g \geq 4$ and $n \geq 5g/2 - 3$ as stated in Theorem 4.

Observe that the condition $n \geq 5g/2 - 3$ is necessary: If $n < 5g/2 - 3$, then G has at most four faces. If G has four faces, then one may use the bound $\left\lceil \frac{g-3}{g-2}n + \frac{2}{g-2} \right\rceil$ which is tight for all g and n with $g \geq 4$ and $n \geq g$. If G has three faces, then either $\chi_f(G) = n-2$ or $\chi_f(G) = n-1$ (it is easy to provide a characterization of the two cases) and if G has less than three faces, then $\chi_f(G) = n-1$ which is clearly optimal.

2. The Lower Bound

In this section, we use the deficit version of Tutte's 1-Factor theorem [3]. A component of a graph is called *odd* or *even* if its number of vertices is odd or even, respectively. Let $q(G)$ denote the number of odd components of a graph G. The graph invariant Δ defined in the next theorem is called the *deficiency* of G:

Theorem 1. *Let G be a multigraph. Then, the size of the largest matching of G is equal to $(n-\Delta)/2$ where $\Delta = \max_{S \subseteq V(G)}(q(G \setminus S) - |S|)$.*

The following lemma has been proved in [14], we include its short proof for the sake of completeness. A *covering* of a graph G is a spanning (but not necessarily connected) subgraph of G with minimum degree at least one.

Lemma 1. *Let G be a plane multigraph with n vertices. Suppose that the dual graph $G^\star$ of G contains a covering subgraph with m edges. Then, $\chi_f(G) \geq n - m$.*

Proof. Let E be the set of m edges of G, which corresponds to the set of edges of the covering of the dual graph $G^\star$. Observe that each face of G is incident with at least one edge of E. Let G' be a graph with a vertex set equal to $V(G)$ and the edge set equal to E. The graph G' consists of at least $n - m$ connected components. Color the vertices of each of its components with the same color and the vertices of different components with different colors. This coloring is clearly a valid coloring of G with at least $n - m$ colors. ∎

A graph G has an *almost perfect matching* if there is a matching of G which omits a single vertex of G. In particular, if G has an almost perfect matching, then G has an odd number of vertices. We use the deficit version of Tutte's theorem to show that highly edge-connected plane multigraphs have coverings with few edges:

Theorem 2. *Let G be a g-edge-connected plane multigraph with f faces and $g \geq 3$. Then, G contains a covering with at most $\left\lfloor \frac{f-2}{g-2} \right\rfloor$ edges unless G has an almost perfect matching.*

Proof. Let n and m be the number of vertices and the number of edges of G, respectively. Let μ be the size of the largest matching of G. By Theorem 1, there exists a set $S \subseteq V(G)$ such that the number of odd components of $G \setminus S$ decreased by $|S|$ is exactly the deficiency $\Delta = n - 2\mu$. Fix such a set S.

If $\Delta = 0$, then G has a perfect matching. A perfect matching of G forms a covering with at most the following number of edges (the inequality below holds since the minimum degree of G is at least g due to the assumption that G is g-edge-connected and the last equality follows from Euler's formula):

$$\frac{n}{2} = \frac{gn - 2n}{2(g-2)} \leq \frac{2m - 2n}{2(g-2)} = \frac{m-n}{g-2} = \frac{f-2}{g-2}.$$

If $\Delta = 1$, then G has an almost perfect matching and the theorem holds vacuously. So, we may assume that $\Delta \geq 2$ in the rest. Furthermore, since G is connected, S is non-empty.

Let α_k be the number of odd components of size k of $G \setminus S$ (this is defined only for odd k's) and β_k the number of even components of size k (for even k's) of $G \setminus S$ and let γ_k be the number of vertices of S with degree k. Let $q = q(G \setminus S) = \sum_{k=1}^{\infty} \alpha_{2k-1}$ be the number of all odd components and $\alpha = \sum_{k=1}^{\infty} (2k-1)\alpha_{2k-1}$ the number of vertices contained in odd components of $G \setminus S$. Similarly, $\beta = \sum_{k=1}^{\infty} 2k\,\beta_{2k}$ denotes the number of vertices contained in even components and $\gamma = \sum_{k=g}^{\infty} \gamma_k$ the size of S.

The number of edges between S and odd components of $G \setminus S$ is at least qg (since G is g-edge-connected) and at most the sum of the degrees of the vertices contained in S. Hence,

$$q\,g \;\leq\; \sum_{k=g}^{\infty} k\,\gamma_k. \tag{2}$$

We now bound the sum of the degrees of the vertices of G using the fact that the minimum degree of G is at least g:

$$(\alpha + \beta)g + \sum_{k=g}^{\infty} k\,\gamma_k \;\leq\; 2m. \tag{3}$$

We combine (2) and (3) to get the following:

$$\begin{aligned}2m &\geq (\alpha+\beta)g + \sum_{k=g}^{\infty}(k-2)\gamma_k + \sum_{k=g}^{\infty} 2\gamma_k \\ &= (\alpha+\beta)g + \sum_{k=g}^{\infty}\frac{k-2}{k}k\gamma_k + 2\gamma \\ &\geq (\alpha+\beta)g + \frac{g-2}{g}\sum_{k=g}^{\infty}k\gamma_k + 2\gamma \\ &\geq (\alpha+\beta)g + q(g-2) + 2\gamma. \qquad (4)\end{aligned}$$

Consider a matching of G of size μ and join each unmatched vertex by an edge to one of its neighbors in order to obtain a covering of G with $M := \mu + \varDelta$ edges. Note that

$$n = \alpha + \beta + \gamma \quad \text{and} \quad \varDelta = q - \gamma.$$

Since $\varDelta = n - 2\mu$, we infer that

$$2M = 2\mu + 2\varDelta = n + \varDelta = \alpha + \beta + \gamma + q - \gamma = \alpha + \beta + q. \qquad (5)$$

On the other hand, we have $m - n + 2 = f$ by Euler's formula. Finally, we combine this with (4) and (5):

$$\begin{aligned}2f &= 2m - 2n + 4 \\ &\geq (\alpha+\beta)g + q(g-2) + 2\gamma - 2(\alpha+\beta+\gamma) + 4 \\ &= (\alpha+\beta+q)(g-2) + 4 \\ &= 2M(g-2) + 4. \qquad (6)\end{aligned}$$

The constructed covering has at most $\left\lfloor \frac{f-2}{g-2} \right\rfloor$ edges due to the last inequality. ∎

Note that if G has an almost perfect matching (which is the case excluded in Theorem 2), then G has a covering with $\frac{n+1}{2}$ edges. Consider just an almost perfect matching with an extra edge joining an unmatched vertex to a (matched) neighbor of it. We are now ready to prove the main result of this section:

Theorem 3. *Let G be a plane graph with n vertices and girth at least g, $4 \leq g \leq n$. Then,*

$$\chi_f(G) \geq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } g = 4, \\ \left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil & \text{if } g \geq 5 \text{ is odd and} \\ \left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil & \text{if } g \geq 6 \text{ is even.} \end{cases}$$

Moreover, if the number of faces of G is even, then:

$$\chi_f(G) \geq \left\lceil \frac{g-3}{g-2}n + \frac{2}{g-2} \right\rceil .$$

Proof. Let f be the number of faces of G and m the number of edges. If G is acyclic, then $\chi_f(G)=n-1$ which is at least the bound claimed in the theorem. Otherwise, we may assume that G is bridgeless: Removing a bridge affects neither the face structure (and hence the coloring problem) nor the girth of G. We also assume that G contains no isolated vertices – otherwise, remove the isolated vertex, color the rest of the graph and add the removed vertex colored with a new color.

Consider now the dual (multi)graph $G^\star$: It has $n^\star=f$ vertices and $f^\star \leq n$ faces (the equality is attained iff G is connected). Note that $G^\star$ is loopless because G has no bridges. Note also that $G^\star$ is g-edge-connected since an edge cut of size $g-1$ or less would correspond to a cycle of length at most $g-1$ contradicting the girth assumption on G. In particular, the minimum degree of $G^\star$ is at least g.

By Theorem 2, $G^\star$ has a covering with at most $\left\lfloor \frac{f^\star-2}{g-2} \right\rfloor \leq \left\lfloor \frac{n-2}{g-2} \right\rfloor$ edges unless $G^\star$ has an almost perfect matching. If $G^\star$ has an almost perfect matching, then there is a covering consisting of an almost perfect matching with an extra edge joining the unmatched vertex of $G^\star$ to a matched vertex. Such a covering has at most at most the following number of edges:

$$\frac{n^\star+1}{2} = \frac{gf-2f}{2(g-2)} + \frac{1}{2} \leq \frac{2m-2f}{2(g-2)} + \frac{1}{2} \leq \frac{n}{g-2} + \frac{g-6}{2(g-2)}.$$

The first inequality holds because the minimum degree of $G^\star$ is at least g and the last inequality is due to Euler's formula. Since $G^\star$ has an almost perfect matching, $n^\star = f$ is odd. If in addition g is odd, then necessarily $2m \geq fg+1$, and hence the bound on the number of edges can be slightly improved:

$$\frac{n^\star+1}{2} = \frac{gf-2f}{2(g-2)} + \frac{1}{2} \leq \frac{m-f}{g-2} - \frac{1}{2(g-2)} + \frac{1}{2} \leq \frac{n}{g-2} + \frac{g-7}{2(g-2)}.$$

Most of the desired inequalities follow now from Lemma 1 and the above bounds for the number of edges in a covering of $G^\star$. If the number of faces of G is even, $G^\star$ cannot have an almost perfect matching and we get:

$$\chi_f(G) \geq n - \left\lfloor \frac{n-2}{g-2} \right\rfloor = \left\lceil \frac{g-3}{g-2}n + \frac{2}{g-2} \right\rceil .$$

Otherwise, if $g \geq 5$ is odd we similarly get:

$$\chi_f(G) \geq n - \left\lfloor \frac{n}{g-2} + \frac{g-7}{2(g-2)} \right\rfloor = \left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil.$$

Finally, if $g \geq 6$ is even we get:

$$\chi_f(G) \geq n - \left\lfloor \frac{n}{g-2} + \frac{g-6}{2(g-2)} \right\rfloor = \left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil.$$

The remaining case is that $g = 4$ and $G^\star$ has an almost perfect matching. Since $g = 4$, it follows that $\frac{g-3}{g-2}n + \frac{2}{g-2} = \frac{n}{2} + 1$. If G is a quadrangulation, then it is properly 2-colorable and thus it follows from (1) that $\chi_f(G) \geq n/\chi(G) + 1 = n/2+1$. Otherwise, the number of faces of G is at most $n-3$ by Euler's formula. Consider a covering comprised of an almost perfect matching plus an extra edge. Such a covering has at most $\frac{n^\star+1}{2} = \frac{f+1}{2} \leq \frac{n-2}{2}$ edges. We infer that:

$$\chi_f(G) \geq n - \left\lfloor \frac{n-2}{2} \right\rfloor = \lceil n/2 \rceil + 1. \qquad \blacksquare$$

The restriction of Theorem 3 to $g = 4$ proves Conjecture 1:

Corollary 1. *If G is an n-vertex triangle-free plane graph, $n \geq 4$, then*

$$\chi_f(G) \geq \lceil n/2 \rceil + 1.$$

3. The Upper Bound

Throughout this section we assume that $g \geq 4$. The construction presented in this section can be found for $g = 4$ in [18]; we extend their technique to larger girths. We introduce an operation which we call a *Y-join*: For a fixed g, set $x = \lfloor g/2 \rfloor - 1$ and $y = \lceil g/2 \rceil - 2$. Note that $x+y+3 = g$. Suppose that G is a plane graph with a face $F = v_1 v_2 \dots v_g$ such that the vertex v_{x+1} is of degree at most 3 and all other vertices incident with F, except maybe v_{g-1} and v_g, are of degree 2. Define $G \oplus_g F$ to be the plane graph obtained from G by adding a path $w_1 w_2 \dots w_{x+y+1}$ to the interior of the face F together with the edges $v_1 w_1$, $v_{y+2} w_{x+1}$ and $v_g w_{x+y+1}$. Such a face F of G is said to be *suitable* for the operation $\oplus_g$. The Y-join operation for $g = 5$ and $g = 6$ are depicted in Figure 1. Observe that the face $w_1 \dots w_{x+y+1} v_g v_1$ of $G \oplus_g F$ is suitable for $\oplus_g$ and so the Y-join operation $\oplus_g$ can be performed iteratively.

In the rest of this section, we call a color of a vertex *unique* if there is exactly one vertex colored with that color.

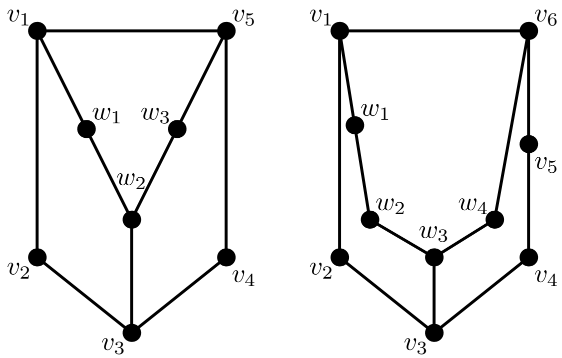

Figure 1. Y-join performed to a face $v_1 \dots v_g$ for $g=5$ and $g=6$.

Lemma 2. *Let G be a plane graph of girth $g \geq 4$ and F a face of G suitable for the operation $\oplus_g$. Then, the girth of $G \oplus_g F$ is g and $\chi_f(G \oplus_g F) \leq \chi_f(G) + g - 3$.*

Proof. We first prove that the girth of $G \oplus_g F$ is g. Assume for the sake of contradiction that there is a cycle C of length smaller than g. The cycle C cannot be facial because the length of the boundary cycle of each of the new faces is exactly $x+y+3=g$. Since the girth of G is g, the cycle C passes through some of the vertices $w_1, \dots, w_{x+y+1}$. Hence, C contains one of the paths $P_1 = v_1 w_1 \dots w_{x+y+1} v_g$, $P_2 = v_1 w_1 \dots w_{x+1} v_{y+2}$ and $P_3 = v_{y+2} w_{x+1} \dots w_{x+y+1} v_g$. The first case is impossible because the path P_1 itself has length $g-1$. In the second case, we may similarly replace P_2 by the path $v_1 v_g v_{g-1} \dots v_{y+2}$ in C to obtain a cycle of length less than g in G. In the third case, we may replace the path P_3 by $v_{y+2} v_{y+1} \dots v_1 v_g$ to obtain a short cycle in G.

We now prove that $\chi_f(G \oplus_g F) \leq \chi_f(G) + g - 3$. Assume that $G \oplus_g F$ has a valid coloring with at least $\chi_f(G) + g - 2$ colors and fix a valid coloring c with exactly $\chi_f(G) + g - 2$ colors. Let Γ be the set of colors used to color $w_1, \dots, w_{x+y+1}$ and Γ_0 be the set of colors used to color the original vertices of G. We distinguish three cases:

- If $|\Gamma \setminus \Gamma_0| \leq g-4$, consider the coloring c restricted to G. This restriction c' uses at least $\chi_f(G)+2$ colors. Note that c' is not a valid coloring of G only if F is rainbow. In that case, recolor v_1 with the color of v_g. Since v_1 is a vertex of degree 2 in G, the new coloring of G is valid and it uses at least $\chi_f(G)+1$ colors which is impossible.
- If $|\Gamma \setminus \Gamma_0| = g-3$, then either $\Gamma \cap \Gamma_0 = \emptyset$ or $|\Gamma \cap \Gamma_0| = 1$.
  In case $\Gamma \cap \Gamma_0 = \emptyset$, no vertex of G is colored by the color of any vertex of $w_1, \dots, w_{x+y+1}$, and precisely two vertices of $w_1, \dots, w_{x+y+1}$ share the

same color. Since c is a valid coloring of $G \oplus_g F$, there must be at least two vertices with the same color among the vertices $v_1, \ldots, v_g$. Thus, the restriction of c to G is a valid coloring of G with $\chi_f(G)+1$ colors.

In case $|\Gamma \cap \Gamma_0| = 1$, all but one of $w_1, \ldots, w_{x+y+1}$ have unique colors. We may assume the vertex with a non-unique color is w_{x+1} (otherwise interchange the colors assigned to w_{x+1} and to the vertex with a non-unique color). Similarly as above we get that at least two vertices of $v_1, \ldots, v_g$ are colored with the same color, and again we have a valid coloring of G with $\chi_f(G)+1$ colors.

- If $|\Gamma \setminus \Gamma_0| = g-2$, then the colors of $w_1, \ldots, w_{x+y+1}$ are unique. Since c is valid, v_1 and v_g have the same color, say A. Recall that v_1 is a vertex of degree two in G.

  If g is even, then $x = y+1$, and so $v_{x+1} = v_{y+2}$ and hence there are two vertices with the same color among the vertices $v_1, \ldots, v_{x+1}$ because of the face $v_1 \ldots v_{x+1} w_{x+1} \ldots w_1$. Consider the coloring of G obtained from $G \oplus_g F$ by assigning a new unique color to v_1. None of the two faces incident with v_1 in G is rainbow (the vertex v_g has the color which the vertex v_1 originally had). Thus, we have obtained a valid coloring of G with $\chi_f(G)+1$ colors, a contradiction.

  Consider now the case that g is odd (in which $x = y$). If the degree of v_{x+1} is two, we proceed as in the case that g is even. Otherwise, the degree of v_{x+1} is three, and the degree of $v_{x+2} = v_{y+2}$ is two in G. Since c is valid, two vertices among $v_1, \ldots, v_{y+2}$ share the same color: There is a vertex v_l with $c(v_l) = A$ for $2 \leq l \leq y+2$ or there are two vertices sharing another color, say B.

  Suppose first that there is a vertex v_l with $c(v_l) = A$. If $l \leq y+1$, then the restriction of c to G with the vertex v_1 recolored with a new unique color is a valid coloring of G with $\chi_f(G)+1$ colors (the vertices v_g and v_l share the same color) which is impossible. If $l = y+2$, interchange the colors assigned to v_{y+1} and v_{y+2} and then proceed as in the case $l \leq y+1$. Observe that the obtained coloring is a valid coloring of G since v_{y+1} has degree three in G.

  Let us now consider the other possibility, i.e., that two distinct vertices of $v_2, \ldots, v_{y+2}$ share the color B. Note that $A \neq B$. Let v_{l_1} and v_{l_2} be two vertices colored with the color B where $2 \leq l_1 < l_2 \leq y+2$. Then, either $l_1 \leq y$ or $l_1 = y+1$.

  Let us first deal with the case that $l_1 \leq y$. Consider a restriction of c to G with the vertex v_1 recolored with a new unique color. If $l_2 = y+2$, then, in addition, interchange the colors of v_{y+1} and v_{y+2}. The obtained coloring is a valid coloring of G because the vertices v_{l_1} and v_{l_2} (or v_{y+1} if $l_2 = y+2$) have the same color. This coloring uses $\chi_f(G)+1$ colors which is a contradiction.

Now, we may assume that $l_1 = y+1$, and so $l_2 = y+2$. Since c is valid, there must be two vertices with the same color among $v_{y+2}, \dots, v_g$. Thus there is a vertex colored by A **or** a vertex colored by B **or** two vertices colored by another color, say C, among $v_{y+3}, \dots, v_{g-1}$.
If there is a vertex v_m (with $y+3 \le m \le g-1$) such that $c(v_m) = A$, consider a restriction of c to G with the vertex v_{y+1} recolored with A and the vertex v_1 recolored with a new unique color. This restriction is a valid coloring of G with $\chi_f(G)+1$ colors (recall that only the vertices v_{y+1}, v_{g-1} and v_g have degrees three or more). If there is a vertex v_m (with $y+3 \le m \le g-1$) such that $c(v_m) = B$ or there are two vertices among $v_{y+3}, \dots, v_{g-1}$ colored with C, then the restriction of c to G with the vertex v_{y+2} recolored to a new unique color is a valid coloring of G with $\chi_f(G)+1$ colors (the colors of v_{l_1} and v_m are still the same), a contradiction. ∎

Lemma 3. *Let G be a graph and G' a graph obtained from G by subdividing a single edge of G. Then, $\chi_f(G') \le \chi_f(G)+1$.*

Proof. Let vw be the subdivided edge of G and u the new vertex. Assume that $\chi_f(G') \ge \chi_f(G)+2$ and let c be a valid coloring of G' with $\chi_f(G)+2$ colors. Let $A = c(v)$ and $B = c(w)$. If the color of the vertex u is not unique, then modify c as follows: Assign the same new color to all the vertices colored by c with A or B (this may decrease the number of colors by one) and then assign to u a new unique color. Neither of the two faces incident with the vertex u is rainbow because the vertices v and w have the same color. Thus the new coloring is a valid coloring with at least $\chi_f(G)+2$ colors.

Hence, we may assume that the color of the vertex u is unique. But then the coloring c restricted to G is a valid coloring of G with $\chi_f(G)+1$ colors which is impossible. ∎

Lemmata 2 and 3 tells us how to build big graphs with large χ_f from small graphs. The next lemma gives us the small graphs to start out from:

Lemma 4. *Let $g \ge 5$. Then, there is a plane graph G_g with n vertices and girth g such that:*

- $n = 5g/2 - 3$ *and* $\chi_f(G_g) \le 5g/2 - 6$ *if g is even;*
- $n = (5g+1)/2 - 3$ *and* $\chi_f(G_g) \le (5g+1)/2 - 6$ *if g is odd.*

In addition, a graph G_g has a face F suitable for the Y-join operation $\oplus_g$.

Proof. Let G_5 and G_6 be the graphs depicted in Figure 2. It is straightforward to verify that the girth of G_5 and G_6 is five and six, respectively. We show that $\chi_f(G_g) \le n-3$ for $g = 5, 6$.

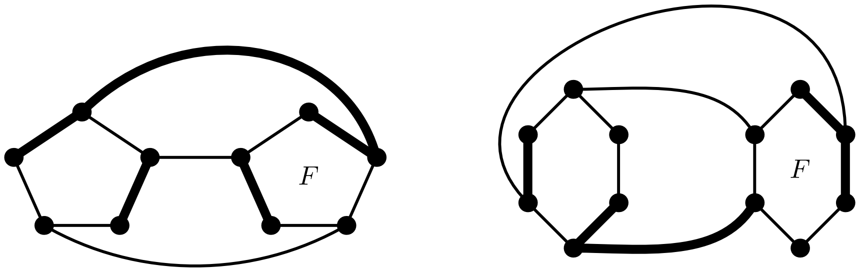

Figure 2. The graphs G_5 and G_6 from Lemma 4.

If $\chi_f(G_g)=n-1$, then there have to be two vertices u and v such that each face is incident with both of them. Such a pair of vertices clearly does not exist. If $\chi_f(G_g)=n-2$, then there is a triple of vertices x, y and z such that each face is incident with at least two of them **or** there are two pairs of vertices such that each face is incident with at least one of these two pairs. In the former case, one of the vertices x, y and z must be incident with four faces because G_g has five faces. Hence G_g must contain a vertex of degree at least four which is not the case. In the latter case, since there are five faces, one of the pairs must be incident with three faces, i.e., there are two vertices u and v such that at least three faces are incident with both of them. This is also not possible. We conclude $\chi_f(G_g)\leq n-3$.

The graphs G_{5+2k} and G_{6+2k} for $k\geq 1$ are obtained from G_5 and G_6, respectively, by subdividing each of the five bold edges in Figure 2 exactly k times. Clearly, G_g for $g=5+2k$ has $(5g+1)/2-3$ vertices and G_g for $g=6+2k$ has $5g/2-3$ vertices. By Lemma 3, $\chi_f(G)\leq n-3$ as desired. The girth of G_{5+2k} is $5+2k$ because each cycle of G_5 contains at least two bold edges (note that G_5 with bold edges deleted is acyclic). Similarly, the girth of G_{6+2k} is $6+2k$. Finally, observe that the face F is suitable for the Y-join operation even after the subdivisions. ∎

We are now ready to prove the main theorem of this section:

Theorem 4. *Let $g\geq 4$ and $n\geq 5g/2-3$. There is a plane graph G of girth g on n vertices such that:*

$$\chi_f(G)=\begin{cases}\left\lceil\frac{n}{2}\right\rceil+1 & \text{if } g=4,\\ \left\lceil\frac{g-3}{g-2}n-\frac{g-7}{2(g-2)}\right\rceil & \text{if } g\geq 5 \text{ is odd and}\\ \left\lceil\frac{g-3}{g-2}n-\frac{g-6}{2(g-2)}\right\rceil & \text{if } g\geq 6 \text{ is even.}\end{cases} \tag{7}$$

In addition, for all g and n, $4 \le g \le n$, there exists a plane graph G with n vertices, with an even number of faces and girth at least g such that $\chi_f(G) = \left\lceil \frac{g-3}{g-2} n + \frac{2}{g-2} \right\rceil$.

Proof. We distinguish the four cases from the statement of the theorem:

If $g=4$, let k and $0 \le l \le 1$ be such that $n = 4+2k+l$. Then, $\lceil \frac{n}{2} \rceil + 1 = 3+k+l$. Consider a graph G obtained from the 4-cycle C_4 by k Y-joins and in addition a subdivision of an edge if $l=1$. The girth of G is clearly at least four. Then, by Lemmata 2 and 3, $\chi_f(G) \le 3+k+l$. The equality follows from Theorem 3.

If $g \ge 5$ is odd, let k and $0 \le l < g–2$ be such that $n = (5g+1)/2–3+k(g–2)+l$. Note that $\left\lceil \frac{g-3}{g-2} n - \frac{g-7}{2(g-2)} \right\rceil = (5g+1)/2 - 6 + k(g-3) + l$. Consider a graph G obtained from G_g of Lemma 4 by k Y-joins and in addition l subdivisions of an edge. The girth of G is clearly at least g. We infer from Lemmata 2, 3 and 4 that $\chi_f(G) \le (5g+1)/2 - 6 + k(g-3) + l$. Finally, the equality holds by Theorem 3.

If $g \ge 6$ is even, let k and $0 \le l < g-2$ be such that $n = 5g/2-3+k(g-2)+l$. Note that $\left\lceil \frac{g-3}{g-2} n - \frac{g-6}{2(g-2)} \right\rceil = 5g/2–6+k(g–3)+l$. Consider a graph G obtained from G_g of Lemma 4 by k Y-joins and in addition l subdivision of an edge. The girth of G is clearly at least g. Again, $\chi_f(G) \le 5g/2 - 6 + k(g-3) + l$ by Lemmata 2, 3, and 4. The equality is due to Theorem 3.

If we want to construct a graph with an even number of faces, we proceed as follows: Let k and $0 \le l < g-2$ be such that $n = g + k(g-2) + l$ and consider a graph G obtained from a cycle of length g by k Y-joins followed by l subdivisions of an edge. Since the number of faces of G is even, we have $\chi_f(G) \ge \left\lceil \frac{g-3}{g-2} n + \frac{2}{g-2} \right\rceil$ by Theorem 3. On the other hand, $\chi_f(G) \le g - 1 + k(g-3) + l = \left\lceil \frac{g-3}{g-2} n + \frac{2}{g-2} \right\rceil$ from $\chi_f(C_g) = g-1$ by Lemmata 2 and 3. ∎

4. Open Problems

We conclude by posing two open problems. The proofs of Lemma 1 and Theorems 2 and 3 can be modified for surfaces of higher genera. The proofs use the fact that each edge cut of the dual graph corresponds to a cycle (this is still true for surfaces of higher genera though the opposite is not the case contrary to the plane). The inequality in Euler's formula for surfaces of higher genera also fits the proof. Let $\mathcal{S}$ be a surface with Euler genus ε. Then, for every graph G embeddable in $\mathcal{S}$ with girth g and with n vertices, we have $\chi_f(G) \ge \left\lceil \frac{g-3}{g-2} n - \frac{g-6+2\varepsilon}{2(g-2)} \right\rceil$ (or a little more if g is odd).

Problem 1. What is the maximum number $f(n,g,\varepsilon)$ such that each graph G on a surface of Euler genus ε, with girth g, and n vertices admits a coloring with $f(n,g,\varepsilon)$ colors with no rainbow faces?

The results of [14] use a weaker assumption than the assumption on the girth, namely the assumption on the lengths of face cycles. This motivates the following question:

Problem 2. What is "the best" function $f(n,g)$ such that for each plane graph G with n vertices and with no face cycle shorter than g, it holds that $\chi_f(G) \geq f(n,g)$?

We do not have examples of plane graphs which would witness that the function $f(n,g)$ differs from the analogous function for the girth.

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